

# Adaptive Multigrid Algorithm for the QCD Dirac-Wilson Operator

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We present a new multigrid solver that is suitable for the Dirac operator in the presence of disordered gauge fields. The key behind the success of the algorithm is an adaptive projection onto the coarse grids that preserves the near null space. The resulting algorithm has weak dependence on the gauge coupling and exhibits mild critical slowing down in the chiral limit. Results are presented for the Wilson Dirac operator of the 2d U(1) Schwinger model.

*The XXV International Symposium on Lattice Field Theory  
July 30-4 August 2007  
Regensburg, Germany*

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## 1. Introduction

The most demanding computational task in lattice QCD simulations consists of the calculation of quark propagators, which are needed both for generating gauge field configurations with the appropriate measure and for the evaluation of most observables. This calculation consists of solving a very large system of linear equations,

$$D(U)\psi = \chi, \quad (1.1)$$

where  $\psi$  is the quark propagator,  $\chi$  is the source term and  $D(U)$  is the discretized the Dirac operator matrix, with elements dependent on the gauge field background  $U$ .

In the language of applied mathematics, Eq. (1.1) is a discretized elliptic partial differential equation (PDE). For definiteness,

$$D_{x,y} = -\frac{1}{2} \sum_{\mu=1}^d ((1 - \gamma_\mu)U_x^\mu \delta_{x+\hat{\mu},y} + (1 + \gamma_\mu)U_{x-\hat{\mu}}^{\mu\dagger} \delta_{x-\hat{\mu},y}) + (2d + m)\delta_{x,y}$$

is the discretized Dirac operator describing a fermion in  $d$  dimensions with mass  $m$  in the Wilson discretization of the Dirac equation. In the full 4 dimensional QCD problem (in volume  $V$ ) the matrices  $\gamma_\mu$  are the  $4 \times 4$  Dirac spin matrices and  $U$  is the  $SU(3)$  gauge field. It is this formulation that we concentrate upon, however, we point out that many of the problems encountered in solving this equation extend to other formulations.

For any realistic QCD calculation the size of the matrix in Eq. (1.1) is too large for a direct solver and iterative Krylov-space methods must be used. As the quark mass is brought towards zero, the condition number of the matrix diverges and hence so does the number of iterations until the desired convergence. This scaling with the mass is commonly referred to as critical slowing down.

It has been known for some time that the multigrid (MG) approach is optimal when solving systems of the form  $Ax = b$ , where  $A$  is the sparse matrix that arises from the discretization of continuum differential equations,  $b$  is a source vector and  $x$  is the desired solution vector. Here discretizations on successively coarser (blocked) grids are used to accelerate the solver and this approach is known to remove critical slowing down [1].

One exception to the above statement is in solving the Dirac operator in lattice QCD: here the nature of the underlying gauge field in the Dirac operator has proven to be especially resistant to MG. Previous attempts at MG solvers have relied on renormalization group arguments to define the coarse grids without realizing why the MG approach succeeds, and this has invariably led to failure as the physically interesting regime is approached [2, 3]<sup>1</sup>. In this work we demonstrate an MG algorithm for the Dirac operator normal equations, i.e., the positive definite operator given by

$$A = D^\dagger D,$$

that is shown to work in all regimes and vastly reduces the notorious critical slowing of the solver as the renormalized fermion mass is brought to zero. We do so in the context of a 2-dimensional

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<sup>1</sup>We note, however, that recent progress has been made in the use of renormalization group to define a coarse Dirac operator, which may render this statement erroneous [4].

system with  $U(1)$  gauge field (Schwinger model). This system captures many of the physical properties (confinement, chiral symmetry breaking, existence of non-trivial topological sectors) of the more complex 4-dimensional QCD.

## 2. Multigrid

The original formulation of MG is best viewed with the example of the free Dirac operator. Multigrid solvers are based on the observation that stationary iterative solvers (e.g., Jacobi, Gauss-Seidel) are only effective at reducing local error components leaving slow to converge, low wave-number components in the error. For the free Dirac operator these slow modes will be geometrically smooth and can be accurately represented on a coarser grid using simple linear averaging. However, on the coarse grid these low wave-number error components become modes of shorter range and so relaxation should be effective at removing them. This process can continue, moving to coarser and coarser grids until the degrees of freedom have been thinned enough to solve the system exactly. The solution is then promoted back to the finest grid using linear interpolation, where at each level relaxation is applied to the correction vector to remove any high wave-number error components that were introduced. This process is known as a V-cycle [1] and can be used as a solver in its own right, or more effectively as a preconditioner for a Krylov method e.g., conjugate gradients (CG).

Before continuing we introduce the notation where the degree of coarseness is represented by the integer  $l$ , where  $l = 1$  represents the finest grid (i.e., where our actual problem is defined) and  $l = L$  is the coarsest grid in an  $L$ -level MG algorithm. The operator used to promote a coarse grid vector on grid  $l + 1$  to the adjacent fine grid  $l$  is known as the prolongator  $P^{(l,l+1)}$ , and the converse operator is the restriction operator  $Q^{(l+1,l)}$  which projects a fine grid vector onto the adjacent coarse grid. Typically the Galerkin definition is used to define the coarse grid operator [1],

$$A^{(l+1)} = Q^{(l+1,l)} A^{(l)} P^{(l,l+1)} = P^{(l,l+1)\dagger} A^{(l)} P^{(l,l+1)}, \quad (2.1)$$

where we have defined the restriction operator as  $Q = P^\dagger$ . This guarantees the coarse grid operator is Hermitian positive definite. That the Galerkin definition is the optimum definition for  $A$  can be found by minimizing the error of the coarse grid corrected solution vector in the  $A$ -norm. Apart from the coarsest level which is an exact solve, each level of the V-cycle is the following

1. Relax on the input vector,  $x^{(l)} = R^{(l)\dagger} b^{(l)}$ , where  $R^{(l)\dagger}$  is a suitable relaxation operator.<sup>2</sup>
2. Restrict the resulting residual to the next coarsest grid,  $r^{(l+1)} = P^{(l,l+1)\dagger} (b^{(l)} - A^{(l)} x^{(l)})$ .
3. Apply the  $L = l + 1$  V-cycle on the coarse residual,  $e^{(l+1)} = V^{(l+1)} r^{(l+1)}$ .
4. Correct the current solution with coarse grid correction,  $x^{(l)} = x^{(l)} + P^{(l,l+1)} e^{(l+1)}$ .
5. Relax on the final residual,  $x^{(l)} = R^{(l)} (b^{(l)} - A^{(l)} x^{(l)})$ .

Written explicitly in terms of operators the  $l^{th}$  level of the V-cycle thus takes the following form

$$V^{(l)} = R^{(l)} + R^{(l)\dagger} + R^{(l)} A^{(l)} R^{(l)\dagger} + \left[ (1 - R^{(l)} A^{(l)}) P^{(l,l+1)} V^{(l+1)} P^{(l,l+1)\dagger} (1 - A^{(l)} R^{(l)\dagger}) \right]. \quad (2.2)$$

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<sup>2</sup>The relaxation operator need not be Hermitian for the entire V-cycle to be Hermitian: the post-relaxation operator need only be the Hermitian conjugate to pre-relaxation.

In this form the Hermiticity of the V-cycle is obvious (a necessary condition if we are to use the V-cycle as a CG preconditioner). The cost of applying the MG V-cycle becomes apparent from this explicit form: on each level we must apply the operator  $A^{(l)}$  a total of  $2\nu + 2$  times for each  $l$ , where  $\nu$  is the number of steps within the relaxation operator.

The problem in the application of the above procedure in the presence of a non-trivial gauge field is that the eigenvectors responsible for slow convergence are no longer low wave-number modes with geometrically smooth variation. They are instead modes that exhibit localized lumps, typically extending over several lattice spacings. An approach that was followed in [2] was to impose Dirichlet boundary conditions along the block boundaries, and to use the low modes of resulting blocked operator to define the prolongator. This approach is bound to produce only a limited advantage since the lumps of the low modes can span between several such blocks, so the blocked operator will not possess this property. Indeed in [2] some acceleration was obtained but critical slowing down was found to return after the correlation length of the pion  $\mu^{-1}$  exceeded the correlation length  $l_\sigma$  of the underlying gauge field.

### 3. Adaptive Multigrid

A breakthrough in the application of multiscale methods to more complex problems occurred with the discovery of adaptive MG techniques [5, 6]. In the adaptive algorithm the MG process itself defines the appropriate prolongator by an iterative procedure which we now concisely describe.

In the first pass, one uses relaxation alone to solve the homogenous problem  $Ae = 0$  with a randomly chosen initial error vector. After a certain number,  $\nu$ , of relaxation steps, the relaxation procedure, which we symbolically represent by

$$e \rightarrow e' = (I - \omega A)^\nu e \equiv (I - \omega D^\dagger D)^\nu e, \quad (3.1)$$

produces an  $e'$  that essentially belongs to the space spanned by the slow modes, so  $e'$  is now used to define a first approximation to the prolongator  $P$ . One blocks the variables of the original lattice into subsets, which we denote by  $S_j$ . From  $e'$  we construct the vectors  $e'_j$ , which are identical to  $e'$  within  $S_j$  and 0 outside  $S_j$ , and the vectors of unit norm  $v_{1j} = e'_j / |e'_j|$ . The extra “1” index in  $v_{1j}$  has been introduced for a discussion that follows. The prolongator  $P^{(1,2)} \equiv P_{i,j}^{(1,2)}$  which maps a vector  $\psi_j^{(2)}$  in the coarse lattice, indexed by  $j$ , to the original lattice, where  $i$  denotes collectively the site, spin and possible internal symmetry indices, is then defined by

$$P_{i,j}^{(1,2)} = v_{1j,i}, \quad (3.2)$$

where we have made explicit the fine lattice indices of  $v_{1j}$ .

There are variations on how to block the fine lattice, i.e., how to define the sets  $S_j$ . In the so called “algebraic adaptive MG” one partitions the fine lattice into subsets on the basis of the magnitude of the matrix elements of  $A$ . Since such matrix elements in lattice gauge theories are typically of uniform magnitude, differing rather in phase or, in a broader sense, in orientation within the space of gauge transformations, we chose instead to partition the lattice geometrically into fixed blocks of neighboring lattice sites, specifically  $4 \times 4$  squares in our study of the Schwinger model. Maintaining a regular lattice on coarse levels will allow more efficient parallel code with exact load balancing.

Another refinement of the technique consists of applying a simple Richardson iteration to the vectors  $v_{1j}$  before defining the prolongator. The choice of damping parameter in this smoothing procedure is chosen to minimize the condition number of the resulting coarse grid operator. The term “smoothed aggregation” is used for this. Thus our overall technique can be referred to as “geometric adaptive smoothly aggregated MG”.

We come now to the crux of the adaptive MG method. We use the prolongator defined above (Eq. (3.2)) to implement a standard MG V-cycle and apply it, like relaxation before, to a randomly chosen error vector. There are two possibilities. Either the V-cycle reduces the error with no sign of critical slowing down or some large error,  $e''$ , survives the cycle. In the first case, of course, one need not proceed: the MG procedure works as is. In the second case, we define another set of vectors  $v_{2j}$  over the coarse lattice by restricting  $e''$  to the subsets  $S_j$ , making the new vectors orthogonal to the vectors  $v_{1j}$  and normalizing them to 1. The smoothed aggregation procedure is now applied to the set  $v_{sj} \equiv (v_{1j}, v_{2j})$ . A new prolongator is defined by projecting over these vectors

$$P_{i,sj}^{(1,2)} = v_{sj,i},$$

where the index  $s$  (taking values 1, 2) can be considered as an intrinsic index over the coarse lattice.

The procedure described in the above paragraph is repeated as necessary, until the repeated application of a V-cycle reduces a random initial error sufficiently without critical slowing down. The method works if critical slowing down is eliminated with a few iterations of the adaptive procedure. If this occurs with  $M$  vector sets, then the coarse lattice will carry  $M$  degrees of freedom per site. As with all MG methods, the procedure is recursive and it can be used to define further coarsenings.

## 4. Results

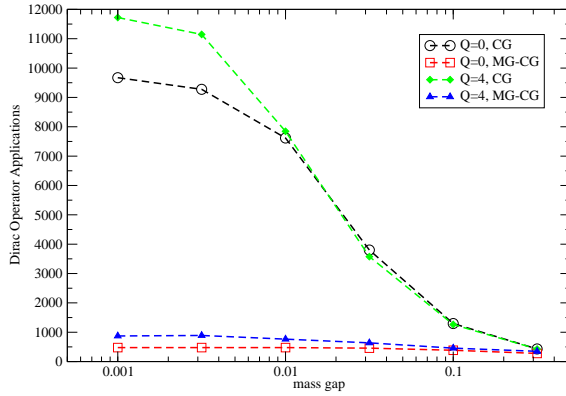
In testing this algorithm for lattice QCD we generated quenched  $U(1)$  gauge field configurations on a  $128 \times 128$  lattice with the standard Wilson gauge field action

$$S = \sum_{x,v < \mu} \beta \operatorname{Re} U_x^{\mu\nu} \equiv \sum_{x,v < \mu} \beta \operatorname{Re} U_x^\mu U_{x+\hat{\mu}}^\nu U_{x+\hat{\nu}}^{\mu\dagger} U_x^{\nu\dagger}$$

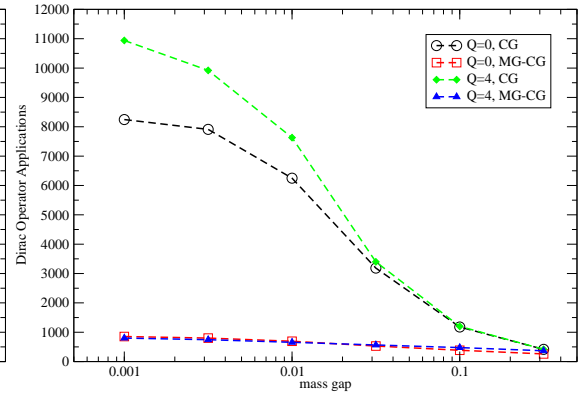
and periodic boundary conditions at  $\beta = 6$  and  $\beta = 10$ . These two values of  $\beta$  define correlation lengths for the gauge field to be  $l_\sigma = 3.30$  and  $l_\sigma = 4.35$  respectively, via the area law for the Wilson loop:  $W \sim \exp[-A/l_\sigma^2]$ . For comparison on these lattices, a fermion mass gap  $\hat{m} = m - m_{\text{crit}} = 0.01$  corresponds to the pseudoscalar meson correlation lengths  $\mu^{-1} = 6.4$  and  $\mu^{-1} = 12.7$  respectively.<sup>3</sup> In the 2-dimensional  $U(1)$  gauge theory, one can identify a gauge invariant topological charge  $\hat{Q}$ , which in the continuum limit is proportional to the quantized magnetic flux flowing through the system. A gauge field with nonzero  $\hat{Q}$  corresponds to a Dirac operator with exactly real eigenvalues and, hence, as the mass gap is brought towards zero the condition number becomes infinite. Thus, it is important to test both trivial ( $\hat{Q} = 0$ ) and non-trivial ( $\hat{Q} \neq 0$ ) gauge field topologies.

We blocked the lattice into  $4 \times 4$  blocks and implemented the adaptive MG procedure described above. We used a degree 2 polynomial smoother for our relaxation procedure, where the

<sup>3</sup>All quantities are expressed in lattice units.



**Figure 1:** Number of Dirac operator applications of CG vs. MG-CG as function of the mass gap at  $\beta = 6$  (point source, relative residual  $|r| = 10^{-14}$ ).



**Figure 2:** Number of Dirac operator applications of CG vs. MG-CG as function of the mass gap at  $\beta = 10$  (point source, relative residual  $|r| = 10^{-14}$ ).

coefficients were chosen by running two iterations of an underrelaxed minimum residual solver ( $\omega = 0.8$ ) and subsequently held fixed (hence, for our choice of smoother  $R = R^\dagger$ ). The coarsening procedure was repeated twice maintaining  $M = 8$  vectors in all coarsenings, down to an  $8 \times 8$  lattice, over which the equations were solved exactly. For each gauge field we performed the set up procedure for the MG preconditioner for the lightest mass parameter only, and reused the vectors for each heavier mass. We used this constructed V-cycle as a preconditioner for CG where the operator defined in Eq. (2.2) is applied at each iteration to the CG direction vector (here on referred to as MG-CG).

If one compares the number of CG iterations needed to achieve convergence with or without MG preconditioning, the gain obtained with the MG method is dramatic: for example, with  $\beta = 6$ ,  $\hat{m} = 0.01$  and  $\hat{Q} = 0$ , it takes 3808 iterations to achieve convergence with a straightforward application of the CG technique, whereas it takes only 26 iterations using MG-CG. However this comparison does not take into account the fact that many more operations per iterations must be performed when applying the MG preconditioner. To achieve a more balanced comparison, in Figs. 1, 2 we plot the total number of applications of  $D$  and  $D^\dagger$  done on the fine lattice. This reflects better the actual cost of the calculations (at each iteration of MG-CG there are 6 applications of  $D^\dagger D$ : 1 application in the outer CG, and 2 pre- and 2 post- coarsening smoothing applications and 1 further application required to form the residual). We do not include the additional cost arising from the coarse lattices since this is expected to be a small overhead, and has not been optimized for our model calculation. The advantage coming from the use of the adaptive MG technique is still very dramatic: critical slowing down, if not totally eliminated, is very substantially reduced and there is no slow down as the pion correlation length exceeds the gauge field correlation length. These results are for point sources, however, we tried a variety of different source vectors for this analysis (e.g., Gaussian noise,  $Z_4$  noise) and found very little dependence of MG-CG performance on the source vector.

From the point of view of computational complexity, one should also take into account the cost of setting up the MG preconditioner, i.e., of constructing the prolongator  $P$ . This cost is however heavily amortized, to the point of being negligible, if, as is often the case, one must apply the solver

to systems with multiple given vectors (for example, solving for all color and spin components of a quark propagator or, in the calculation of disconnected diagrams where,  $O(1000)$  inverses are required to estimate the trace of the inverse Dirac operator).

## 5. Conclusion

Our results, albeit for now limited to a 2-dimensional example, provide a clear indication that adaptive MG can be made to work with the lattice Dirac operator. What appears to be at the root of its success is that, although the modes responsible for slow convergence of the Dirac solver on a fine lattice are not low wavenumber excitations, like in the free case, their span can be well approximated by a set of vectors of limited dimensionality on the blocks that define the coarse lattice.<sup>4</sup> Earlier attempts [2, 3] failed to eliminate critical slowing down when the pseudoscalar length exceeded the disorder length of the gauge field:  $\mu^{-1} > l_\sigma$ . Adaptive MG finds the coarse subspaces through the iterative application of the method itself. It is of course crucial that the approximation to the space of slow modes can be achieved with a small number of vectors on the individual blocks, otherwise the application of the method would not be cost effective. But this appears to be the case in the examples we studied and, if the results hold true in general, adaptive MG has the potential of substantially speeding up lattice QCD simulations as the increase of available computational power leads one to consider ever larger lattices.

Current research is focussed on applying this adaptive MG algorithm to the Dirac operator directly, as opposed to the normal equations. Here we are motivated to do so because of the reduced condition number and the increased sparsity of the operator. There are added complications because the Dirac operator isn't Hermitian (this requires that the restriction and prolongation operators are defined using left and right vectors respectively), however, initial progress is extremely promising. The application of the method to large 4-dimensional systems is in progress,

Acknowledgments. This research was supported in part under NSF grant PHY-0427646 and DOE grants DE-FG02-91ER40676 and DE-FC02-06ER41440.

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<sup>4</sup>The observation that the space of slow modes may be of limited span is also at the root of a method recently proposed by Lüscher in Ref. [7], although the technique there is quite different from the one we follow.